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Decoherence of an Open System under Continuous Quantum Measurement of Energy

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Abstract We study continuous quantum measurements (CQM) of energy of an open quantum system by Lindblad master equation. It turns out that the time-dependence of decoherence is identified. We conclude that the CQM of energy accelerate quantum decoherence.

Keywords Continuous quantum measurements · Decoherence · Master equation · Density matrix elements

The simplest description of a measurement in quantum physics was provided by von Neumann [1]. When quantum measurement was first introduced to quantum mechanics, it was invariably treated by ignoring the time the measurement takes. However, it is not sufficient to describe continuously monitored system. Usually, one wants to understand what happens to the system while the measurement takes place continuously. This is the subject of continuous quantum measurement [2]. Recently, continuous or repeated measurements of quantum system has been actively discussed due to its implication in feedback control [3], metrology [4–7], quantum information [8], quantum computing [9–11] and its importance in understanding the quantum to classical transition [12–15].

When a quantum system is measured, its state is changed which may be described as decoherence. Decoherence is a basic idea in the theory of CQM which plays a decisive role in the dynamics of a system subject to repeated or continuous quantum measurement. The interest in decoherence is widespread, due to the fact that it is the main limiting factor for quantum information processing [16].

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Usually, the evolution of a continuously measured system is described by master equation. In Markovian approximation, the state of the system is presented by a density matrix, which satisfies the time–differential equation. This approach is a non-selective one, i.e., without taking the concrete readout of the measurement into account. The most general form of master equation was derived by Lindblad [17]. The system they discussed in [18] only subjected to quantum measurements of energy. The authors did not take the effect of its surroundings into account. Since no system is really isolated from its surroundings, it is more practical to study continuous measurements of an open system. In this paper, we will discuss CQM of energy for an open system. We will study the decoherence of the continuously measured system and discuss the effect the energy measurement in detail by Lindblad master equation

Von Neumann described two fundamentally different types of interventions by which a quantum state can change in his famous textbook [1]. One is the effects of a measurement; the other is the automatic changes associated with evolution via the Schrödinger equation. The case of open system evolution is the second "intervention". Under appropriate conditions, open system evolves in a way that reflects the effect of measurements being made on the system. It has been recognized that the dynamics of the system in the presence of the measurement undergoes modifications, with respect to the closed system dynamics, that can be taken into account by means of an effective master equation or a Schrödinger equation for mixed and pure states, respectively. In order to describe the evolution of an open system, one can work with the reduced density operator $\hat{\rho}(t)$ of the system. The equation of motion of the reduced density operator is obtained by tracing out the degrees of freedom of the environment from the density matrix operator of the entire open system. The unitary evolution of $\hat{\rho}(t)$ for the isolated system is modified to be irreversible due to the interaction with the environment. In the Markov limitation, $\hat{\rho}(t)$ can be described completely by the Lindblad master equation as follows [17, 19]

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[H, \hat{\rho}] + \frac{1}{2}\sum_j \{[L_j\hat{\rho}, L_j^+] + [L_j, \hat{\rho}L_j^+]\}, \tag{1}$$

where $\hat{\rho}(t)$ is the density operator of the system, H is the Hamiltonian of the system and L_j is the so-called Lindblad operators that are supposed to model the effect of the environment on the system. Equation (1) describes the general case of a quantum open system. The evolution of a system subjected to a measurement process corresponds to the particular case where the environment is the measurement apparatus and the Lindblad operators are proportional to the measured quantities. $L_j = \sqrt{\gamma}\hat{A}$, $(j = 1, 2, 3, \ldots)$, $\hat{A}$ is an observable operator and γ is the coupling constant.

Let us consider an open system S which coupled to a bath of bosons B (harmonic oscillators). The Hamiltonian of this compound system can be written as

$$\hat{H} = \hat{H}_S + \sum_k \omega_k a_k^+ a_k + \hat{\Lambda}_S \sum_k (g_k a_k^+ + g_k^* a_k), \tag{2}$$

where a_k and a_k^+ are the bosonic creation and annihilation operators, $\hat{H}_S$ is the Hamiltonian of the open system S. The second term in (2) is the Hamiltonian $\hat{H}_B$ of an infinite number of harmonic oscillators B labeled by the subscript k and the last term is the interaction Hamiltonian $\hat{H}_I$. g_k and g_k^* are the coupling constants.

In this work, we propose to study a special case of no energy exchange between the system, i.e., $[\hat{H}_S, \hat{H}] = 0$. That's to say $[\Lambda_{S,} \hat{H}_S] = 0$. We can select $|n\rangle$ as their eigenstates, i.e., $\hat{H}_S|n\rangle = E_n|n\rangle$, $\hat{\Lambda}_S|n\rangle = \lambda_n|n\rangle$.

Our aim is to obtain the density matrix of the system S and B. It has the following matrix elements in the preferred basis:

$$\begin{aligned}\frac{d\rho_{mn}}{dt} &= \left[-\frac{i}{\hbar}(E_m - E_n) + \frac{\gamma}{2}(2E_m E_n - E_m^2 - E_n^2)\right]\rho_{mn} \\ &\quad - \frac{i}{\hbar}\left[\lambda_m \sum_k \left(g_k a_k^+ + g_k^* a_k\right) + \sum_k \omega_k a_k^+ a_k\right]\rho_{mn} \\ &\quad + \frac{i}{\hbar}\rho_{mn}\left[\lambda_n \sum_k (g_k a_k^+ + g_k^* a_k) + \sum_k \omega_k a_k^+ a_k\right], \end{aligned} \tag{3}$$

where $\langle m|\hat{\rho}|n\rangle = \rho_{mn}$. From (3), we can get the solution of $\rho_{mn}(t)$ easily

$$\begin{aligned}\rho_{mn}(t) &= \exp\left\{\left[-\frac{i}{\hbar}(E_m - E_n) + \frac{\gamma}{2}(2E_m E_n - E_m^2 - E_n^2)\right]t\right\} \\ &\quad \times \exp\left\{-\frac{i}{\hbar}\left[\lambda_m \sum_k (g_k a_k^+ + g_k^* a_k) + \sum_k \omega_k a_k^+ a_k\right]t\right\}\rho_{mn}(0) \\ &\quad \times \exp\left\{\frac{i}{\hbar}\left[\lambda_n \sum_k (g_k a_k^+ + g_k^* a_k) + \sum_k \omega_k a_k^+ a_k\right]t\right\}. \end{aligned} \tag{4}$$

We assume initially the quantum system is in a pure or mixed state described by the density matrix $\rho^S(0)$, not entangled with the bath. For the bath, we assume that each oscillator is independently thermalized at temperature T with the density matrix θ_k. The total system-plus-bath density matrix will be

$$\rho(0) = \rho^s(0)\prod_k \theta_k, \tag{5}$$

here $\theta_k = Z_k^{-1} e^{-\beta\omega_k a_k^+ a_k}$, $Z_k \equiv (1 - e^{-\beta\omega_k})^{-1}$, Z_k is the partition function for the oscillator k. The quantity β is defined as $\beta = \frac{1}{\kappa T}$. In order to obtain the reduced density matrix element ρ_{mn}^s of the system S, we can trace over the freedom of its surroundings. The reduced density matrix of the open system S has the following matrix elements

$$\rho_{mn}^s(t) = \mathrm{Tr}_{\mathrm{B}}(\rho_{mn}(t)). \tag{6}$$

Dima Mozyrsky and Vladimir Privman have obtained the result of trace exactly in [20]:

$$\begin{aligned} &\mathrm{tr}\Bigg(\exp\left\{-\frac{i}{\hbar}\left[\lambda_m \sum_k (g_k a_k^+ + g_k^* a_k) + \sum_k \omega_k a_k^+ a_k\right]t\right\}\rho_{mn}(0) \\ &\quad \times \exp\left\{\frac{i}{\hbar}\left[\lambda_n \sum_k (g_k a_k^+ + g_k^* a_k) + \sum_k \omega_k a_k^+ a_k\right]t\right\}\Bigg), \\ &= \prod_k \exp(-\omega_k^{-2}|g_k|^2 P_{mn,k}), \end{aligned}$$

where

$$P_{mn,k} = 2(\lambda_m - \lambda_n)^2 \sin^2\frac{\omega_k t}{2}\coth\frac{\omega_k\beta}{2} + i(\lambda_m^2 - \lambda_n^2)(\sin\omega_k t - \omega_k t). \tag{7}$$

From (5), (6) and (7), the reduced density matrix element ρ^s_{mn} of the system S can be rewritten as

$$\rho^s_{mn}(t) = \exp\left\{\left[-\frac{i}{\hbar}(E_m - E_n) - \frac{\gamma}{2}(E_m - -E_n)^2\right]t\right\} \times \prod_k \exp(-\omega_k^{-2}|g_k|^2 P_{mn,k})\rho^s(0). \tag{8}$$

Equation (8) suggests that the diagonal elements in the reduced density matrix remain unchanged and the effect of the measurement and the interaction with its surroundings are to diagonalize the density matrix after enough time. Decoherence is clearly controlled by the interaction with the heat bath and measurement. The eigenvalues of Λ_S and H_S determine the rate of decoherence, while the type of the bath and the coupling controls the form of the function $P_{mn,k}$. The states with equal eigenvalues λ_m will not remain entangled except energies E_m are the same.

We add $H_R = \Lambda_S \sum_k \omega_k^{-1}|g_k|^2$ in (2) in our calculation to modify the imaginary part of $P_{mn,k}$. The modified expression of $P_{mn,k}$ is $P_{mn,k} = 2(\lambda_m - \lambda_n)^2 \sin^2\frac{\omega_k t}{2}\coth\frac{\omega_k\beta}{2} + i(\lambda_m^2 - \lambda_n^2)\sin\omega_k t$. Usually, Hamiltonian is real, i.e., $g_k = g_k^*$. Defining $R(t) = \mathrm{Tr}(\rho_S^2)$ as a measure of coherence [21] and setting the system starts from a pure state ($\rho_S(0) = \sum_n P_n|n\rangle\sum_m P_m\langle m|$), we get

$$R_N(t) = \sum_{mn}|P_n|^2|P_m|^2\exp(-2\Gamma(t) - \gamma(E_m - E_n)^2 t), \tag{9}$$

where $\Gamma(t) = 2(\lambda_m - \lambda_n)^2\sum_k \omega_k^{-2}|g_k|^2\sin^2\frac{\omega_k t}{2}\coth\frac{\omega_k\beta}{2}$.

For the bosons bath we introduce the Ohmic spectral density functions, i.e., $\rho(\omega) = \eta\frac{\omega}{|g(\omega)|^2}e^{-\frac{\omega}{\omega_c}}$ with ω_c being the high frequency cutoff and we have:

$$\Gamma(t) = \left(\frac{\eta}{2}\ln(1 + (\omega_c t)^2) + \eta\ln\prod_{k=1}^{\infty}\left(1 + \left(\frac{\omega_c t}{1 + k\beta\omega_c}\right)^2\right)\right)(\lambda_m - \lambda_n)^2. \tag{10}$$

At the condition of $\omega_c\beta \gg 1$ the expression above can be rewritten as:

$$\Gamma(t) = \left(\frac{\eta}{2}\ln(1 + (\omega_c t)^2) + \eta\ln\left(\frac{\sinh t/\tau}{t/\tau}\right)\right)(\lambda_m - \lambda_n)^2, \tag{11}$$

here $\tau = \beta/\pi$. Then we have:

$$R_N(t) = \sum_{mn}|P_n|^2|P_m|^2 \mathrm{e}^{(-\gamma(E_m - E_n)^2 t)}(1 + (\omega_c t)^2)^{-\eta(\lambda_m - \lambda_n)^2}\left[\frac{\sinh t/\tau}{t/\tau}\right]^{-2\eta(\lambda_m - \lambda_n)^2}. \tag{12}$$

In the experimentally accessible domain of time i.e., $\omega_c t \gg 1$. At finite temperature, we obtain

$$R_N(t) = \sum_{mn}|P_n|^2|P_m|^2 \mathrm{e}^{(-\gamma(E_m - E_n)^2 t)}(\omega_c t)^{-2\eta(\lambda_m - \lambda_n)^2} e^{-\frac{2\eta}{\tau}(\lambda_m - \lambda_n)^2}. \tag{13}$$

And in the Markov limitation (with $\omega_c t = 1$), we have

$$R_N(t) = \sum_{mn}|P_n|^2|P_m|^2 \mathrm{e}^{(-\frac{\gamma'}{\beta}(\lambda_m - \lambda_n)^2 - \gamma(E_m - E_n)^2 t)}. \tag{14}$$

Now, we find the relaxation of nondiagonal elements following the exponential law. The approximate result derived from the master equation in [22, 23] is

$$R_N(t) = \sum_{mn} |P_n|^2 |P_m|^2 \mathrm{e}^{(-\frac{\gamma'}{\beta}(\lambda_m - \lambda_n)^2 t)}. \tag{15}$$

In the Markov limitation, we find that for $\gamma = 0$, (14) is equal to (15). Thus one can see that CQM of energy and interaction with environment are reduced to affect the system separately. The effect of CQM of energy and interaction with environment is to diagonalize the density matrix after enough time while the diagonal elements in the reduced density matrix remain unchanged. Comparing (14) with (15), we conclude that the CQM of energy accelerate quantum decoherence.

If the system S is immersed in the TLS reservoir with non-demolition couplings, the total Hamiltonian has the form

$$H_T = H_S + H_B + H_I, \tag{16}$$

where H_S is the same as in (2) , but $H_B = \sum_k \omega_k \sigma_{z_k}$ and $H_I = \Lambda_S \sum_k c_k \sigma_{x_k}$. We still assume that $\rho(0) = \rho^s(0) \prod_k \theta_k$. Note that according to (1) the reduced density matrix becomes

$$\begin{aligned} \rho^s_{mn}(t) &= \mathrm{Tr_B}(\rho_{mn}(t)) \\ &= \exp\left\{ -\frac{i}{\hbar}(E_m - E_n)t - \frac{\gamma}{2}(E_m - E_n)^2 t \right\} \rho^s_{mn}(0) \\ &\quad \times \mathrm{tr_B}\left[\prod_k \exp\left(-\frac{i}{\hbar} h_{km} t \right) \theta(0) \exp\left(\frac{i}{\hbar} h_{kn} t \right) \right] \\ &= \exp\left\{ -\frac{i}{\hbar}(E_m - E_n)t - \frac{\gamma}{2}(E_m - E_n)^2 t \right\} \rho^s_{mn}(0) \\ &\quad \times \prod_k \mathrm{tr_k}\left[\exp\left(-\frac{i}{\hbar} h_{km} t \right) \theta_\mathrm{k}(0) \exp\left(\frac{i}{\hbar} h_{kn} t \right) \right], \end{aligned} \tag{17}$$

where $h_{kn} = \lambda_n \sum_k c_k \sigma_{x_k} + \sum_k \omega_k \sigma_{z_k}$. Making use of the identity

$$\exp[i(\xi\sigma_z + \mu\sigma_x)] = \cos\sqrt{\xi^2 + \mu^2} + \frac{i \sin\sqrt{\xi^2 + \mu^2}}{\sqrt{\xi^2 + \mu^2}}(\xi\sigma_z + \mu\sigma_x),$$

we obtain [24]

$$\begin{aligned} &\mathrm{tr_k}\left[\exp\left(-\frac{i}{\hbar} h_{km} t \right) \theta_\mathrm{k}(0) \exp\left(\frac{i}{\hbar} h_{kn} t \right) \right] \\ &\quad = \cos[\omega'_k(E_m)t] \cos[\omega'_k(E_n)t] \\ &\qquad + \frac{\sin[\omega'_k(E_m)t] \sin[\omega'_k(E_n)t]}{\omega'_k(E_m)\omega'_k(E_n)} [\omega_k^2 + \lambda(E_m)\lambda(E_n)|c_k|^2], \end{aligned} \tag{18}$$

where $\omega'_k = \sqrt{\omega_k^2 + |c_k|^2 \lambda^2}$. To get an explicit expression for the reduced density matrix element we should assume that the couplings are very weak, i.e., $c_k \approx 0$. Thus

$$\mathrm{tr_k}\left[\exp\left(-\frac{i}{\hbar} h_{km} t \right) \rho_{mn}(0) \exp\left(\frac{i}{\hbar} h_{kn} t \right) \right] \approx 1 - [\lambda(E_m) - \lambda(E_n)]^2 \frac{c_k^2}{\omega_k^2} \sin^2(\omega_k^2 t).$$

Inserting into (17), we have

$$\begin{aligned}\rho^s_{mn}(t) &= \mathrm{Tr}_{\mathrm{B}}(\rho_{mn}(t))\\ &\approx \exp\left\{-\frac{i}{\hbar}(E_m - E_n)t - \frac{\gamma}{2}(E_m - E_n)^2 t\right\}\rho^s_{mn}(0)\\ &\quad\times \prod_k \left\{1 - [\lambda(E_m) - \lambda(E_n)]^2 \frac{c_k^2}{\omega_k^2}\sin^2(\omega_k^2 t)\right\}\\ &= \exp\left\{-\frac{i}{\hbar}(E_m - E_n)t - \frac{\gamma}{2}(E_m - E_n)^2 t\right\}\rho^s_{mn}(0)\\ &\quad\times (1 + 4\omega_c^2 t^2)^{-\eta[\lambda(E_m)-\lambda(E_n)]^2/4}. \end{aligned} \tag{19}$$

Here we used the Ohmic spectral density distribution in the derivation. As shown above, a prominent consequence of the energy measurement of the quantum non-demolition coupling is that the diagonal elements remain unchanged during revolution. The reduced density matrix elements are independent of temperature which represents the very different between the harmonic oscillator bath and the TLS bath.

In summary, we study continuous quantum measurements (CQM) for a general quantum system interacting with environment. We find that in the Markov limitation, the diagonal elements in the reduced density matrix remains unchanged and the effect of the measurement and the interaction of its surroundings will diagonalize the density matrix after enough time.

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